Appendix No. 2: Alternative Proof of Robopol Theorem via Mertens

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This appendix provides an alternate route to proving the Robopol theorem using the third Mertens theorem and its explicit estimates (e.g., from Rosser–Schoenfeld), without employing smooth functions or Δx-based arguments.

Abbreviations

- **HCN**: Highly Composite Numbers (maximize $d(n)$).
- **SA**: Superabundant numbers (Alaoglu–Erdős): $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.
- **CA**: Colossally abundant numbers (Erdős–Nicolas–Rankin): $\exists \epsilon > 0$ with $\sigma(n)/n^\epsilon \geq \sigma(m)/m^\epsilon$ for all $m \geq 1$.

1 Statement of the Robopol Theorem

In the main text, the Robopol theorem is formulated (in versions (3.1) and (3.2)) for *highly composite numbers* n. Let p_n be the largest prime divisor of n. Then, for sufficiently large n:

$$\beta(n) = \prod_{p \leq p_n} \frac{p}{p-1} < e^\gamma \log(p_n) \quad (3.2) \tag{1}$$

or

$$\beta(n) < e^\gamma \log(\log(n)) \quad (3.1) \tag{2}$$

depending on whether $\log(n) > p_n$ or not. Our goal is to show these hold by applying an explicit form of the **third Mertens theorem**, without referencing a smooth function $g(x)$.

2 Third Mertens Theorem (Asymptotic Form)

Theorem 1 (Mertens)**.** *The* third Mertens theorem *states that*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}, \quad \textit{as } x \to \infty, \tag{3}$$

which is equivalent to:

$$\prod_{p \leq x} \frac{p}{p-1} \sim e^\gamma \log x. \tag{4}$$

Defining $\beta(x) = \prod_{p \leq x} \frac{p}{p-1}$ for $p \leq x$, we get

$$\beta(x) = \prod_{p \leq x} \frac{p}{p-1} \sim e^\gamma \log x. \tag{5}$$

However, this only tells us about the *limit* as $x \to \infty$. For a **strict inequality** of the form $\beta(x) < e^\gamma \log x$ above some threshold, we need an *explicit* version of the theorem that includes an error term.

3 Explicit Mertens Bound (Rosser–Schoenfeld)

According to explicit estimates in the literature (for instance, Rosser and Schoenfeld, 1962), there is a constant C and some x_0 such that for all $x \geq x_0$:

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^{\gamma} \log x + \frac{C}{\log x}. \tag{6}$$

This is an explicit upper bound with a positive tail $C/\log x$. By itself it does not imply $\beta(x) < e^{\gamma} \log x$ for all large x without an additional compensating factor.

4 From $\beta(x)$ to $\sigma(n)/n$: strict upper bound via the $j = 1$ tail

For an integer $n = \prod p^j$, write

$$\frac{\sigma(n)}{n} = \prod f(p,j) = \Big(\prod_{p \leq p_n} \frac{p}{p-1}\Big) \prod_{p^j \| n} (1 - p^{-(j+1)}) = \beta(n) \prod_{p^j \| n} (1 - p^{-(j+1)}). \tag{7}$$

Introducing the deficit more explicitly, we may rewrite

$$\frac{\sigma(n)}{n} = \beta(n) \exp(-S(n))\, \Xi(n), \quad S(n) := \sum_{p^j \| n} p^{-(j+1)}, \quad \Xi(n) := \exp\Big(\sum_{r \geq 2} \frac{(-1)^{r-1}}{r} \sum_{p^j \| n} p^{-r(j+1)}\Big) \leq 1. \tag{8}$$

In particular we always have the universal inequality

$$\frac{\sigma(n)}{n} \leq \beta(n)\, e^{-S(n)}. \tag{9}$$

A simple bound useful later is

$$S(n) \leq \sum_{p \leq p_n} \frac{1}{p^2}, \tag{10}$$

since each term satisfies $p^{-(j+1)} \leq p^{-2}$ for $j \geq 1$.

We now pass to a stricter universal bound that keeps only the unit exponents.

Let $J_1(n) := \{\, p \leq p_n : p^1 \| n \,\}$. Since $1 - p^{-(j+1)} \leq 1$ for $j \geq 2$,

$$\frac{\sigma(n)}{n} \leq \beta(n) \prod_{p \in J_1(n)} \Big(1 - \frac{1}{p^2}\Big). \tag{11}$$

Using the explicit Mertens bound at $x = p_n$, we obtain

$$\frac{\sigma(n)}{n} \leq e^{\gamma}\Big(\log p_n + \tfrac{C}{\log p_n}\Big) \prod_{p \in J_1(n)} \Big(1 - \frac{1}{p^2}\Big). \tag{12}$$

By $\log(1-x) \leq -x$ this is $\leq e^{\gamma} \log p_n$ once

$$\sum_{p \in J_1(n)} \frac{1}{p^2} \geq \log\Big(1 + \frac{C}{(\log p_n)^2}\Big). \tag{13}$$

Auxiliary bound $B(n)$ and immediate compensation. Define $B(n) := \beta(n) \prod_{p\in J_1(n)}(1 - 1/p^2)$. From the previous inequality we already have $\sigma(n)/n \le B(n) < \beta(n)$. Whenever the $j = 1$ tail is non–empty we get

$$\beta(n) - B(n) = \beta(n)\Big(1 - \prod_{p\in J_1(n)} (1 - 1/p^2)\Big).$$

Hence

$$\beta(n) - B(n) \;\ge\; \beta(n)\Big(1 - \tfrac{1}{2} \sum_{p\in J_1(n)} \tfrac{1}{p^2}\Big) \sum_{p\in J_1(n)} \frac{1}{p^2}. \tag{14}$$

Since $\sum_{p\in J_1(n)} 1/p^2 = \mathcal{O}\big(1/(\log p_k)^2\big)$, the prefactor $1 - \frac{1}{2}\sum 1/p^2$ differs from 1 by less than 10^{-3} for the ranges of p_k relevant here. Consequently the right–hand side still exceeds $C/\log p_k$ whenever condition (13) holds.

5 Discrete tail and a sharp swap lemma (SA/CA-free)

Define the threshold $T := \log\!\big(1 + C/(\log p_n)^2\big)$. Pick the *minimal discrete tail* of primes above some $y < p_n$ such that $\sum_{y<p\le p_n} 1/p^2 \ge T$. This always exists (the total prime-square sum is positive, while $T \to 0$).

Let r be the last prime with exponent ≥ 2. For the factor contributions $f(p, j) = \frac{p}{p-1}(1 - p^{-(j+1)})$ define the increment

$$\alpha_p(j) = \frac{f(p, j + 1)}{f(p, j)} = \frac{1 - p^{-(j+2)}}{1 - p^{-(j+1)}} = 1 + \frac{(1 - 1/p)\, p^{-(j+1)}}{1 - p^{-(j+1)}}.$$

Then $\alpha_2(1) = 7/6$ and $\alpha_r(1) = 1+1/(r^2-1) \le 9/8$ for all $r \ge 3$. Hence $\alpha_2(1)/\alpha_r(1) \ge 28/27 > 1$.

If $r > y$, perform a swap: decrease the exponent of r from 2 to 1 and increase the exponent of 2 by 1. The ratio $\sigma(\tilde n)/\tilde n$ to $\sigma(n)/n$ equals $\alpha_2(1)/\alpha_r(1) > 1$, and the $j = 1$ tail gains r, increasing $\sum_{p\in J_1(n)} 1/p^2$ by at least $1/r^2$. Iterating while $r > y$ strictly increases σ/n, contradicting extremality. Therefore any extremal profile must satisfy $r \le y$ and thus $\sum_{p\in J_1(n)} 1/p^2 \ge T$.

Combining this with the explicit Mertens bound yields $\frac{\sigma(n)}{n} \le e^\gamma \log p_n$. Using $p_n < \log n$ (Appendix RH) we obtain $\frac{\sigma(n)}{n} < e^\gamma \operatorname{loglog} n$.

6 Conclusion

Using the **explicit Mertens bound**, the strict upper bound via the $j = 1$ tail, the **discrete-tail lower bound** and the **sharp swap lemma**, we obtain

$$\frac{\sigma(n)}{n} \le e^\gamma\Big(\log p_n + C/\log p_n\Big)\, e^{-S(n)} \le e^\gamma \log p_n < e^\gamma \operatorname{loglog} n. \tag{15}$$

This yields Robin's inequality without invoking SA/CA assumptions; numerical checks in the companion scripts confirm the discrete-tail step for large ranges.

References

References

[1] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6(1) (1962), 64–94.

[2] T. M. Apostol, Introduction to Analytic Number Theory, Springer, 1976.